Interpretation of Differentials

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We frequently solve geometrical and physical problems by obtaining an approximate expression for differential dP in terms of differential dQ and then integrating dP to obtain P. We assume that the expression for P is exact even though we used an approximate formula for dP. This is justified by saying that the differentials are infinitely small quantities. For example, when we derive an expression for the area of a circular disc (see example 1) we set $dA = 2\pi r dr$ which is an approximate expression when the diffentials are interpreted as real numbers. In this article we try to define a method for computing P so that we don't need approximate expressions in the derivation.

Theorem 1. *Let $a, b \in \mathbb{R}$ and $a < b$. Let f be a function from $[a, b]$ into $\mathbb{R}$ and define $\Delta f = f(x + \Delta x) - f(x)$ where $x \in \mathbb{R}$ and $\Delta x \in \mathbb{R}\backslash\{0\}$ and $x, x + \Delta x \in [a, b]$. Suppose that*

$$\Delta f = g(x)\Delta x + h(x, \Delta x)$$

where x and Δx are defined as before. Suppose also that g is Riemann integrable and

$$\lim_{\Delta x \to 0} \frac{h(x, \Delta x)}{\Delta x} = 0 \tag{1}$$

for all $x \in [a, b]$. Now $df = g(x)dx$ and

$$f(x) - f(a) = \int_a^x g(t)dt.$$

Proof. This is a direct consequence of the definition of differentiability and Fundamental Theory of Calculus. □

The condition (1) can be weakened to

$$\lim_{\Delta x \to 0+} \frac{h(x, \Delta x)}{\Delta x} = 0. \tag{2}$$

Theorem 2. *A sufficient condition for equation (2) is that there exist $S, C \in \mathbb{R}_+$ so that*

$$|h(x, \Delta x)| < C|\Delta x|^2$$

for all $x, x + \Delta x \in [a, b]$ and $0 < \Delta x < S$.

Proof. We have

$$\left|\frac{h(x, \Delta x)}{\Delta x}\right| < C|\Delta x| \to 0$$

as $\Delta x \to 0+$. □

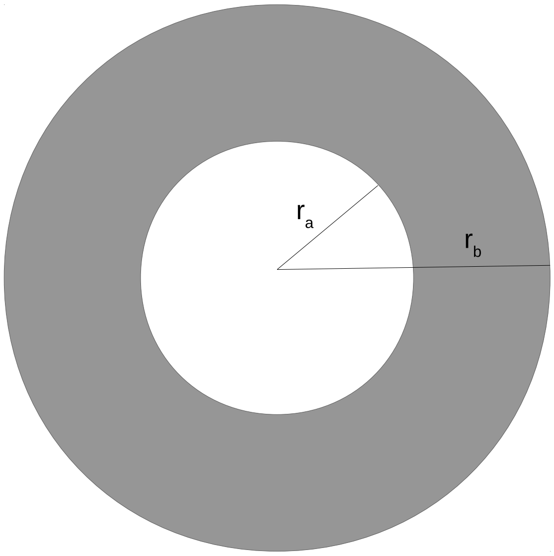

Figure 1: The area determined in example 1.

Example 1. *Derive an expression for the area of a disc whose inner radius is* r_a *and outer radius* r_b*.*

Solution: Define ΔA to be the area of a disc with inner radius r and width Δr. We have

$$2\pi r\Delta r \leq \Delta A \leq 2\pi(r + \Delta r)\Delta r$$

By setting $g(r) := 2\pi r$ and $h(r, \Delta r) := 2\pi(\Delta r)^2$ we get $A = \pi r_b^2 - \pi r_a^2$ by Theorems 1 and 2.

Example 2. *Suppose that a particle is moving under influence of a constant force* $F = ma$ *for time* T *and the particle is initially at rest. Derive an expression for the kinetic energy of the particle. Assume that the work done by a constant force* F *is* $W = Fs$ *where* s *is the distance that the particle moves in the direction of the force. Assume also that the kinetic energy of a particle at rest is* 0*.*

Solution: We define Δs to be the distance that the particle moves in the time interval $[t, t + \Delta t]$. We have $v = at$,

$$at\Delta t \leq \Delta s \leq a(t + \Delta t)\Delta t,$$

and

$$a(t + \Delta t)\Delta t = at\Delta t + a(\Delta t)^2.$$

Set $g(t) := at$ and $h(t, \Delta t) := a(\Delta t)^2$ and it follows from Theorems 1 and 2 that the distance that the particle moves in time T is

$$s = \int_0^T at dt = \frac{1}{2}aT^2$$

By setting $v_f = aT$ we obtain

$$E_k = W = \frac{1}{2}FaT^2 = \frac{1}{2}ma^2T^2 = \frac{1}{2}mv_f^2.$$

Alternative Solution: Assume that the particle moves distance Δs in time Δt. Define $\Delta W := F\Delta s$. Now the acceleration $a = \Delta v/\Delta t$ is a constant and we have

$$\Delta W = ma\Delta s = m\Delta v\frac{\Delta s}{\Delta t} \tag{3}$$

Let $v_{\min}$ and $v_{\max}$ be the minimum and maximum velocities of the particle. We now have

$$v_{\min} \leq \frac{\Delta s}{\Delta t} \leq v_{\max}.$$

If $\Delta v \geq 0$ we get

$$mv_{\min}\Delta v \leq \Delta W \leq mv_{\max}\Delta v,$$

which is equivalent to

$$mv\Delta v \leq \Delta W \leq m(v + \Delta v)\Delta v.$$

If $\Delta v < 0$ we have

$$v + \Delta v \leq \frac{\Delta s}{\Delta t} \leq v,$$

from which it follows that

$$mv\Delta v \leq \Delta W \leq m(v + \Delta v)\Delta v.$$

Define

$$h(v, \Delta v) := \Delta W - mv\Delta v.$$

Now

$$0 \leq h(v, \Delta v) \leq m\Delta v^2.$$

By setting $g(v) = mv$ and assuming that the kinetic energy is zero when $v = 0$ it follows from Theorems 1 and 2 that

$$E_k = W = \frac{1}{2}mv^2.$$